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ON A GENERALIZATION OF ALBERT'S THEOREM

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ABSTRACT

It is proved by A. A. Albert that in an ordered division ring, any element algebraic over the center is central. In this paper, we shall investigate the following problem. Let D be an ordered division ring. Suppose every element in D is left algebraic over a maximal subfield K. Does it follow that $D = K$? We prove that the answers are affirmative in some cases.

1. Introduction

Let D be a division ring. A subset $P \subset D$ is called an ordering on D, if (i) $P + P \subset P$, (ii) $P \cdot P \subset P$, (iii) $P \cup (-P) = D$, (iv) $P \cap (-P) = \{0\}$. In this case, we say (D, P) is an ordered division ring, and write $a >_P b$ if $a - b \in P \setminus \{0\}$. (For convenience, we shall simply write $a > b$ instead of $a >_P b$ if there is no confusion of ordering concerned. Moreover we shall denote $\max(x, -x)$ by $|x|$.) It was proved by A. A. Albert [A_1] that in an ordered division ring, any element algebraic over the center is in fact central. This theorem is of major importance in the theory of ordered division rings. It plays a key role in the proof of many results, for examples, see [A_2] and [Le_2]. In 1982, it was generalized by V. Tamhankar [Tam] in case of ordered rings though there is a slight restriction involved. However, his proof employs some combinatorial ideas. For a proof based on valuation theory, see [Le_1]. In this paper, we attempt to generalize Albert's result in a different direction.

The first naive attempt is to replace the center by a maximal subfield. Of course, we need to be careful with what we mean by an element being 'algebraic' over a maximal subfield.

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Let t be a central indeterminate over D. We write $D[t]$ for the ring of left polynomials over D. Also, for any $f(t) = \Sigma_{i=0}^{n} a_i t^i$ in $D[t]$ and α in D, we define $f(\alpha) = \Sigma_{i=0}^{n} a_i \alpha^i$.

DEFINITION 1.1. Let D, $D[t]$ be as above, L be a division subring of D and α be an element in D. If there exists a nonzero $f(t) \in L[t] \subset D[t]$ such that $f(\alpha) = 0$, then we say α is a root of $f(t)$ and α is left algebraic over L. Moreover if every element in D is left algebraic over L, then we say D is left algebraic over L.

REMARK. To simplify language, we shall drop the adjective "left" in the following.

With the above definition, we can now state the first question. If an element x is algebraic over a maximal subfield K in an ordered division ring D, does it follow that $x \in K$? This turns out to be false in general, as we shall show below.

LEMMA 1.2. *Let D be a division ring with a maximal subfield K. Suppose the characteristic of D is* 0. *Then the following statements are equivalent*:

(i) *There exist $x \in D$, $a \in K$ such that $xa = ax + 1$.*

(ii) *There exist $y \in D$, $a \in K$ such that $y^2 - 2ay + a^2 = 0$.*

Furthermore, if x, a, are as in (i), *then x is not algebraic over K.*

PROOF. By writing $y = x^{-1} + a$, we easily see that (i) and (ii) are equivalent. Now suppose x is algebraic over K and $h(t) = \Sigma_{i=0}^{n} b_i t^i$ is the minimal polynomial of x over K. By induction, we have $x^r a = ax^r + rx^{r-1}$. So,

$$\begin{aligned} 0 &= (b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0)a \\ &= b_n(ax^n + nx^{n-1}) + b_{n-1}(ax^{n-1} + (n-1)x^{n-2}) + b_1(ax + 1) + b_0 a \\ &= a(b_n x^n + \cdots + b_0) + nb_n x^{n-1} + \cdots + b_1 \\ &= nb_n x^{n-1} + \cdots + b_1. \end{aligned}$$

This is a contradiction as x satisfies a nonzero polynomial over K with degree less than $\deg h$. ∎

EXAMPLE. Let $R := \mathbf{R}[u, v]$ be the Weyl algebra defined by the relation $uv = vu + 1$ and D' be its ring of quotients. It is well known that D' is a division ring which can also be ordered [D: p. 318]. If we let K' be a maximal subfield in D' containing v, then by the previous lemma, we see that $u^{-1} + v$ is algebraic over K'. Clearly $u^{-1} + v$ does not lie in K'.

In view of the above example, we modify our question as follows.

QUESTION. Let K be a maximal subfield of an ordered division ring D. If D is algebraic over K, does it imply $D = K$?

Firstly, let us discuss a trivial case. If $(K, P \cap K)$ is archimedean, then it can be easily shown that (D, P) is also archimedean and therefore a field. Naturally, we then consider the case when K is $k(x)$ or a finite extension of $k(x)$, where $(k, k \cap P)$ is archimedean and x is transcendental over k. Note that we do not assume k lying inside the center of D. These cases are already nontrivial. We shall prove in Section 3 that D is actually a field under the above assumption on K and D. In fact, the method developed there is applicable in a more general context. In Section 4, we shall concentrate on the case when K is real closed. However, we need to assume that the transcendence degree tr.d. $K/Z(D)$ is finite. In particular, we know that the answer is again affirmative in case tr.d. $K/\mathbf{Q}$ is finite.

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2. Preliminary results

As before, we let D be an ordered division ring with a maximal subfield K. Even though we have shown above that an element algebraic over K does not necessarily lie in K, we can still say something about its minimal polynomial over K. The following observation is due to Prof. T. Y. Lam.

LEMMA 2.1. *Let D' be a division subring of D such that $Z(Z(D')) = D'$ (i.e. the double centralizer of D' is itself). Suppose $\alpha \in D$ is algebraic over D' and $f(t)$ is its monic minimal polynomial over D'. Then*

$$f(t) = (t - x_1 \alpha x_1^{-1}) \cdots (t - x_n \alpha x_n^{-1}) \qquad \text{for some } x_i \in D,$$

and every root of $f(t)$ is conjugate to α. In particular, the above is true when $D' = K$.

PROOF. This follows easily from [W: Lemma 4] or [C: Proposition 3.3.7]. ∎

Note that the above lemma is true even if D cannot be ordered. When D can be ordered, the way that $f(t)$ can be factorized is extremely important, because we can then compare the coefficients of $f(t)$ with α. As a corollary, we record the following generalization of Albert's Theorem [A₁].

PROPOSITION 2.2. *Let* α, $f(t)$, D' *be as defined in Lemma* 2.1. *Suppose* $\alpha \notin D'$ *and* $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$. *Then* $a_i \notin Z(D)$ *for* $i = 0, 1, \ldots, n-1$.

PROOF. Without loss of generality, we may assume that $\alpha > 0$.

By Lemma 2.1, there exist $x_j \in D^* = D \setminus \{0\}$, $j = 1, 2, \ldots, n$ such that for $i = 0, 1, \ldots, n-1$

$$a_i = \sum_{j_1 < j_2 < \cdots < j_{n-i}} x_{j_1} \alpha x_{j_1}^{-1} \cdots x_{j_{n-1}} \alpha x_{j_{n-i}}^{-1}.$$

Let

$$x_j \alpha x_j^{-1} = \max_s x_s \alpha x_s^{-1}, \qquad x_k \alpha x_k^{-1} = \min_s x_s \alpha x_s^{-1}.$$

So we have

$$\binom{n}{i} (x_k \alpha x_k^{-1})^i \leqq a_i \leqq \binom{n}{i} (x_s \alpha x_s^{-1})^i.$$

If $a_i \in Z(D)$, then we have $\alpha^i \leqq a_i / \binom{n}{i} \leqq \alpha^i$. Hence $\alpha^i = a_i / \binom{n}{i} \in Z(D)$. This clearly implies $\alpha \in Z(D)$. But this is impossible as by assumption $D' = Z(Z(D')) \supset Z(D)$ and $\alpha \notin D'$. ∎

To conclude this section, we recall some basic definitions and results of valuations which we shall apply in Section 3. For references, we refer the readers to [S], [T₁] and [T₂].

A valuation on a division ring L is a mapping $v : L \to G \cup \{\infty\}$, where G is a totally ordered group (written additively though not necessarily abelian), such that for all $a, b \in L$,

(i) $v(a) = \infty$ if and only if $a = 0$,

(ii) $v(ab) = v(a) + v(b)$,

(iii) $v(a + b) \geqq \min\{v(a), v(b)\}$.

Also, we let $R_v := \{a \in L : v(a) \geqq 0\}$, the valuation ring of v; $I_v := \{a \in L : v(a) > 0\}$, the unique maximal left ideal and maximal right ideal of R_v; $\bar{L}_v := R_v / I_v$, the residue division ring of v and $\pi_v : R_v \twoheadrightarrow \bar{L}_v$, the natural projection from R_v to $\bar{L}_v$.

Next, we define the notion of compatibility between orderings and valuations on a division ring.

DEFINITION 2.3. Let $v: L \to G \cup \{\infty\}$ be a valuation and Q be an ordering on L. We say v is compatible with the ordering Q if for any $a, b \in Q \setminus \{0\}$, $a - b \in Q \setminus \{0\}$ implies $v(a) \leqq v(b)$ in G.

LEMMA 2.4 (T_1: Lemma 3.4). *Let (L, Q) be an ordered division ring. Then v is compatible with Q if and only if $1 + I_v \subset Q$.*

As a consequence of the above lemma, we see that any valuation v compatible with an ordering Q on a division ring L induces an ordering

$$\bar{Q}_v := \{a + I_v : a \in R_v \cap Q\} \qquad \text{on } \bar{L}_v \qquad [T_2\text{: Section 0}].$$

3. Ordered division rings with certain kind of compatible valuations

Let α be an ordinal number and $A_\alpha := \Pi_{\gamma<\alpha} \mathbf{Z}$. For any element $a = \Pi_{\gamma<\alpha} a_\gamma \in A_\alpha$, we define $\mathrm{supp}(a) = \{\gamma : a_\gamma \neq 0\}$ and say $\mathrm{supp}(a)$ is well ordered if every subset in $\mathrm{supp}(a)$ contains a maximal element. Furthermore, let $\Gamma_\alpha := \{a \in A_\alpha : \mathrm{supp}(a) \text{ is well ordered}\}$. Obviously, Γ_α is an abelian group. Next, we define a mapping S on Γ_α such that $S(a) = 0$ if $a = 0$; otherwise, we define $S(a) = a_\gamma$ where γ is the largest ordinal in $\mathrm{supp}(a)$. Clearly $Q_\alpha := \{x \in \Gamma_\alpha : S(x) \geqq 0\}$ is an ordering on Γ_α. In the literature, we say $(\Gamma_\alpha, Q_\alpha)$ is the lexicographic product of $\mathbf{Z}$. For a reference, see [F: Chapter II, §7].

Suppose α, β are two ordinals with $\beta < \alpha$. We can regard Γ_β as an isolated subgroup (i.e. for any $a \in \Gamma_\alpha$, if there exists $b \in \Gamma_\beta$ such that $|a| < |b|$, then $a \in \Gamma_\beta$) of Γ_α. Thus, $\Gamma_\alpha / \Gamma_\beta$ is also an ordered abelian group. We shall denote the natural projection from Γ_α onto $\Gamma_\alpha / \Gamma_\beta$ by $\eta_{\alpha,\beta}$.

Let us fix the following throughout this section. (D, P) is an ordered division ring which is algebraic over a maximal subfield K and $\phi: D \to \Gamma \cup \{\infty\}$ is a valuation compatible with P. Here, we also assume an ordering on Γ is fixed. In this section, we shall prove that if Γ is order isomorphic to $(\Gamma_\alpha, Q_\alpha)$, where α is an ordinal, then D is a field. The first step is to show the following.

PROPSOITION 3.1. *Let (D, P), K, ϕ, Γ be as above. Then:*
(i) *$\phi(D^*) = \phi(K^*)$. In particular we can assume Γ is abelian.*
(ii) *$\bar{D}_\phi$ is algebraic over $\bar{K}_\phi := \pi_\phi(R_\phi \cap K)$. Furthermore, if $\bar{D}_\phi$ is a field, then $\bar{D}_\phi = \bar{K}_\phi$.*

PROOF. Let $x \in D \setminus K$ and $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in K[t]$ be a minimal monic polynomial of x over K. Clearly, we may assume $x > 0$, $n \geqq 2$ and write a_n for 1. By Lemma 2.1, we have

$$f(t)=(t-y_1xy_1^{-1})\cdots(t-y_nxy_n^{-1}) \qquad \text{for some } y_i\in D.$$

As $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0=0$, there exist $0\leqq i,\ j\leqq n$ such that $a_i, a_j\neq 0$ and $\phi(a_ix^i)=\phi(a_jx^j)=\min\{\phi(a_sx^s):0\leqq s\leqq n\}\neq\infty$. Therefore, $(j-i)\phi(x)=\phi(a_i)-\phi(a_j)$. Since this is true for any $x\in D\setminus K$, we conclude that $\phi(D^*)$ is abelian and lying inside the divisible hull of $\phi(K^*)$.

Now, observe that $a_{n-1}=\Sigma_{i=1}^n y_ixy_i^{-1}$ and all $y_ixy_i^{-1}$'s are of the same sign. Thus, there exist $1\leqq i,\ j\leqq n$ such that $ny_ixy_i^{-1}\leqq a_{n-1}\leqq ny_jxy_j^{-1}$. As ϕ is compatible with P and $\phi(D^*)$ is abelian, we conclude $\phi(a_{n-1})=\phi(x)$. It follows that $\phi(D^*)=\phi(K^*)$.

Suppose $x\in R_\phi$. It is clear that for $i=0,\ldots,n-1$, we have

$$a_i=\sum_{j_1<j_2<\cdots<j_{n-i}} y_{j_1}xy_{j_1}^{-1}\cdots y_{j_{n-i}}xy_{j_{n-i}}^{-1}.$$

Since $\phi(D^*)$ is abelian, $\phi(y_{j_1}xy_{j_1}^{-1}\cdots y_{j_{n-i}}xy_{j_{n-i}}^{-1})=(n-i)\phi(x)\geqq 0$. Hence we have $\phi(a_i)\geqq 0$. Thus, all $\bar{a}_i$'s are well defined and $\bar{f}(t)=t^n+\bar{a}_{n-1}+\cdots+\bar{a}_0\in\bar{K}_\phi[t]$. Obviously, $\bar{f}(\bar{x})=0$ in $\bar{D}_\phi$. Hence $\bar{D}_\phi$ is algebraic over $\bar{K}_\phi$.

Suppose $\bar{D}_\phi$ is a field. $\phi(D^*)=\phi(K^*)$ implies that for any $y\in D^*$, there exists $b\in K$ such that $yb^{-1}\in U_\phi$. Hence for any $c\in R_\phi\cap K$, $\overline{ycy^{-1}}=\overline{yb^{-1}cby^{-1}}=\overline{yb^{-1}}\,\bar{c}\,\overline{yb^{-1}}^{\,-1}=\bar{c}$. In particular, we have $\phi(y_ia_{n-1}y_i^{-1}-a_{n-1})>0$ for $1\leqq i\leqq n$. Thus,

$$\phi\left(\sum_{i=1}^n y_i(nx-a_{n-1})y_i^{-1}\right)=\phi\left(\sum_{i=1}^n (a_{n-1}-y_ia_{n-1}y_i^{-1})\right)>0.$$

Since all $y_i(nx-a_{n-1})y_i^{-1}$'s are of the same sign, by compatibility of ϕ, we have $\phi(nx-a_{n-1})>0$. This clearly implies $n\bar{x}=\bar{a}_{n-1}$. So, we conclude $\bar{D}_\phi=\bar{K}_\phi$. ∎

REMARK. Note that our proof depends heavily on the compatibility of ϕ and (D,P). In case D is not ordered, say D is finite dimensional over its center F, it is well known that $|\phi(D):\phi(F)|$ could be equal to $|D:F|$ [TW].

COROLLARY 3.2. *If* $\Gamma\cong\mathbf{Z}$, *then* $D=K$.

PROOF. By considering the valued topology T_ϕ on D, it is easy to see from the above proposition that K is dense in D. Therefore $\phi(axa^{-1}-x)\geqq n$ for any $n\in\mathbf{N}$, $a\in K^*$ and $x\in D$. Hence D is a field and must be K.

REMARK. (i) In the above corollary, all we need are $\bar{D}_\phi=\bar{K}_\phi$ and $\phi(K^*)$ equal to $\phi(D^*)=\mathbf{Z}$. They are crucial as we can construct an ordered division ring

(D, P) (not algebraic over any maximal subfield though) with a compatible valuation ϕ such that

(a) $\phi(D^*) = \phi(K^*) \cong \mathbf{Z}$ and $\bar{D}_\phi$ is isomorphic to any subfield k of $\mathbf{R}$ with tr.d. $k/\mathbf{Q} \geqq 1$ or

(b) $\bar{D}_\phi = \bar{K}_\phi$ and $\phi(D) = \phi(K)$ is a dense subgroup of $\mathbf{Q}$.

For details of the construction, we refer the readers to [Sch: Chapter 4] and [Le_3].

(ii) It is not clear if the above corollary is true if Γ is not discrete. But in case Γ is $\mathbf{Q}$ and K is real closed with tr.d. $K/\mathbf{Q}$ being finite, then again $D = K$. For its proof, we refer the readers to Section 4.

We shall improve the above result to the case when Γ is order isomorphic to $(\Gamma_\alpha, Q_\alpha)$ for some ordinal α. Before that, we record a lemma.

LEMMA 3.3. *Let $v: L \to \Delta \cup \{\infty\}$ be a valuation. Suppose Δ' is a convex normal subgroup of Δ and $\eta: \Delta \to \Delta/\Delta'$ is the natural projection. Then:*

(i) *$v': L \xrightarrow{v} \Delta \cup \infty \xrightarrow{\eta} (\Delta/\Delta') \cup \infty$ is a valuation coarser than v.*

(ii) *If we define $v/v': \bar{D}_{v'} \to \Delta' \cup \{\infty\}$ such that*

$$v/v'(a + I_{v'}) = v(a) \quad \forall a \in U_{v'} \quad \text{and} \quad v/v'(I_{v'}) = \infty,$$

then v/v' is a valuation with residue division ring isomorphic to $\bar{D}_v$.

(iii) *If v is compatible with an ordering Q on L, then v' is compatible with Q and v/v' is compatible with $\bar{Q}_{v'}$ on $\bar{D}_{v'}$.*

PROOF. (i) is obvious. To see that v/v' is a valuation on $\bar{D}_{v'}$, see [T_1: p. 17]. Note that $U_{v'} = \{a \in L : v(a) \in \Delta'\}$. Hence $v/v'(\bar{D}_{v'}) \subset \Delta'$. As for the last statement of (ii), just observe that

$$R_{v/v'} = \{a + I_{v'} : a \in L \text{ and } v(a) \geqq 0\}$$

and

$$I_{v/v'} = \{a + I_{v'} : a \in L \text{ and } v(a) > 0\}.$$

Obviously, $R_{v/v'}/I_{v/v'} \cong R_v/I_v = \bar{D}_v$. Finally, (iii) is a consequence of [T_1: Chapter 1, Theorem 3.10]. ∎

THEOREM 3.4. *Let (D, P), K, Γ be as before. If Γ is order isomorphic to a $(\Gamma_\alpha, Q_\alpha)$ for some ordinal α and if $\bar{D}_\phi$ is a field, then D is also a field.*

PROOF. We shall proceed by induction on α. If $\alpha = \{0\}$, then $\Gamma_\alpha = \{0\}$ and so the valuation is trivial. Therefore $D \cong \bar{D}_\phi$ is a field. If $\alpha = \{0, 1\}$, then $\Gamma_\alpha \cong \mathbf{Z}$. So by Corollary 3.2, D is a field.

Now suppose the proposition is true for all ordinals $<\alpha$. Let $\beta<\alpha$ and

$$\phi_\beta : D \xrightarrow{\phi} \Gamma_\alpha \cup \{\infty\} \xrightarrow{\eta_{\alpha,\beta}} (\Gamma_\alpha/\Gamma_\beta) \cup \{\infty\}.$$

By Lemma 3.3(i), we see that ϕ_β is a valuationon D.

CLAIM. $\bar{D}_{\phi_\beta}$ is a field.

By Lemma 3.3 again, we see that ϕ/ϕ_β is compatible with $\bar{P}_{\phi_\beta}$ on $\bar{D}_{\phi_\beta}$. Moreover, its residue division ring is isomorphic to $\bar{D}_\phi$ which is a field. By Lemma 3.1(ii) and induction, we get the desired claim.

We now consider two different situations.

Case (i). α is a successor of some $\beta<\alpha$ (i.e. $\alpha=\beta+1$).

Note that $\Gamma_\alpha/\Gamma_\beta \cong \mathbf{Z}$ and $\bar{D}_{\phi_\beta}$ is a field by the above claim. So by induction again, D is a field.

Case (ii). α is a limit ordinal.

Let $x, y \in D^*$, if $xy-yx \neq 0$ then $\phi(xy-yx)$, $\phi(x), \phi(y) \in \Gamma_\gamma$ for some $\gamma<\alpha$. Hence we have $x, y, xy-yx \in U_{\phi_\beta}$ for some $\gamma<\beta<\alpha$. Note that β exists as α is a limit ordinal. Thus, $\bar{x}, \bar{y}, \overline{xy-yx} \neq 0$ in $\bar{D}_{\phi_\beta}$. This contradicts the claim that $\bar{D}_{\phi_\beta}$ is field. ∎

Before we state the next corollary, let us recall that the natural valuation v_Q of an ordered division ring (L, Q) is the one induced by the valuation ring $\{a \in D : |a| < n \text{ for some } n \in \mathbf{N}\}$. It is well known that v_Q is compatible with Q and (L_{v_Q}, Q_{v_Q}) is an archimedean field [T_1: Theorem 3.5, 3.6].

COROLLARY 3.5. *Let v be the natural valuation on (D, P). If $v(D^*)$ is order isomorphic to a subgroup of $(\Gamma_\alpha, Q_\alpha)$ for some ordinal α then D is a field.*

PROOF. This follows immediately from the above theorem and the well known fact that $\bar{D}_v$ is isomorphic to a subfield of $\mathbf{R}$.

COROLLARY 3.6. *Let (K', P') be an ordered field, and v be its natural valuation. If $v(K'^*)$ is order isomorphic to a subgroup of $(\Gamma_\alpha, Q_\alpha)$ for some ordinal α, then there exists no noncommutative ordered division ring (D', P'') satisfying the following conditions:*

(i) *K' is a maximal subfield of D' and $P' = P'' \cap K$,*

(ii) *D' is algebraic over K'.*

In particular, the above conclusion is true if $K' = k(x)$ or a finite extension of $k(x)$ where $(k, k \cap P')$ is archimedean.

PROOF. It is clear that if v' is the natural valuation of (D, P), then $v'|_K = v$. But by Lemma 3.1, $v(D') = v(K')$. Thus, the corollary follows immediately from the above corollary. As for the last statement, the given assumption implies $v(K^*)$ is $\{0\}$ or isomorphic to $\mathbf{Z}$. ∎

4. Real closed maximal subfields

Let (D, P), K be as in Section 3. In this section, we shall only consider the case when K is real closed. We shall prove that if tr.d. $K/Z(D)$ is finite, then $D = K$.

Since K is real closed, every irreducible factor of a nonconstant polynomial in $K[t]$ is either linear or quadratic. Obviously, all the irreducible factors will split into products of linear factors in $K[i][t]$ where $i = \sqrt{-1}$. Unfortunately, $\sqrt{-1}$ does not exist in D. Thus we must enlarge D so as to contain i. The following construction might be known, but we are unable to find any reference.

LEMMA 4.1. *Let $D' := D \oplus Di$ with i commuting with every element in D. Then D' is a division ring. Moreover, if $a + bi \neq 0$, then $(a + bi)^{-1} = (1 - a^{-1}bi)(a + ba^{-1}b)^{-1}$.*

PROOF. Let t be a central indeterminate. Note that $D' \cong D[t]/(t^2 - 1)$ is a division ring if $t^2 - 1$ is irreducible over D, iff D does not contain a square root of -1. As D is ordered, it does not contain a square root of -1. Hence D' is a division ring. Lastly, by a straightforward calculation, it can be easily shown that $(a + bi)^{-1} = (1 - a^{-1}bi)(a + ba^{-1}b)^{-1}$. ∎

There exists an automorphism "bar" of order 2 on D. Namely, for all $x + yi \in D'$, we define $\overline{x + yi} := x - yi$. Clearly $K' := K[i]$ is algebraically closed and is a maximal subfield of D'. Therefore for any $a \in D'^*$, $aK'a^{-1} \subset K'$ iff $aK'a^{-1} = K'$. Let $N(K') = \{a \in D'^* : aK'a^{-1} = K'\}$. Obviously it is a multiplicative subgroup of D'^* containing K'^*. Furthermore, as "bar" is an automorphism and $\overline{K'} = K'$, it follows that $\overline{N(K')} = N(K')$. Our first goal is to prove that $N(K') = K'^*$. While doing so, we need a theorem due to Gerstenhaber and Yang [GY].

THEOREM 4.2. *Let D be a division ring and K be a real closed field in D. If $[D : K]_l < \infty$ (i.e. D is a finite dimensional left vector space over K), then either $D = K$, $K(\sqrt{-1})$ or a quaternion algebra over a real closed field. In particular, if $\sqrt{-1} \in Z(D)$, then $D = K(\sqrt{-1})$ is a field.*

Our first step is to show the following:

LEMMA 4.3. *If $\omega \in N(K')\backslash K'$, then ω is not algebraic over K'. In particular, $N(K') \cap D^* = K^*$.*

PROOF. Suppose ω is algebraic over K'. Let D_1 be the division subring generated by K' and ω. Since $\omega \in N(K')$, we see that $D_1 = K' + K'\omega + \cdots + K'\omega^r$ for some $r \in \mathbf{N}$. Thus $[D_1 : K']_l < \infty$. This implies $[D_1 : K]_l < \infty$. By Theorem 4.2, $D_1 = K'$, which is impossible as $\omega \notin K'$. The last statement is obvious as by assumption D is algebraic over K. ■

PROPOSITION 4.4. $N(K') = K'^*$.

PROOF. Let $x + yi \in N(K')\backslash K'$. By Lemma 4.3, both x, y are nonzero. Since $\overline{N(K')} = N(K')$, we have $x - yi \in N(K')\backslash K'$. As $N(K')$ is a multiplicative group,

$$\omega = a + bi := (x - yi)(x + yi)^{-1} \in N(K').$$

Clearly,

$$\bar{\omega}\omega = (x + yi)(x - yi)^{-1}(x - yi)(x + yi)^{-1} = 1.$$

Case (i). $\omega \in K'$. But then

$$x = \tfrac{1}{2}[(x - yi) + (x + yi)] = \tfrac{1}{2}(\omega + 1)\cdot(x + yi) \in N(K') \cap D^*.$$

By Lemma 4.3, we have $x \in K$. Similarly, we get $y \in K$ and therefore $x + yi \in K'$. This is a contradiction.

Case (ii). $\omega \notin K'$. Now, $2a = \omega + \bar{\omega} \in D$ is algebraic over K. Thus there exists a monic minimal polynomial $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in K[t]$ such that

$$(\omega + \bar{\omega})^n + a_{n-1}(\omega + \bar{\omega})^{n-1} + \cdots + a_0 = 0.$$

So after multiplying the above equation by ω^n on the right, we get

$$(\omega^2 + 1)^n + a_{n-1}(\omega^2 + 1)^{n-1}\omega + \cdots + a_0\omega^n = 0.$$

Let $g(t) = (t^2 + 1)^n + a_{n-1}(t^2 + 1)t + \cdots + a_0 t^n$. Then obviously, $0 \neq g(t) \in K[t]$ and $g(\omega) = 0$. This contradicts Lemma 4.3. ■

For any element $a \in D'$, let us define $C'(a) = \{y \in D' : ya = ay\}$. Suppose $a, b \in K'$, and $a = xbx^{-1}$. If $C'(a) = K'$, then we have $xK'x^{-1} \subset xC'(b)x^{-1} = K'$. Therefore, x is in $N(K')$ which equals K'^* by the above proposition. It follows that $a = b$. In other words, if $C'(a) = K'$, then a is the only conjugate of itself lying inside K'. This is a very important observation which leads us to prove the following lemma:

LEMMA 4.5. *Let D', D, K be as before. If $a \in K \setminus Z(D)$, then $C'(a) \neq K'$.*

PROOF. Assume on the contrary, there exists $a \in K \setminus Z(D)$ with $C'(a) = K'$. Since $a \notin Z(D)$, there exists $x \in D$ such that $b := xax^{-1} \notin K$. By assumption on D, there exists a minimal monic polynomial $f(t) \in K'[t]$ such that $f(b) = 0$. Note that as $b \notin K'$, $n \geqq 2$. Now, as K' is algebraically closed, $f(t) = (t - a_1) \cdots (t - a_n) \in K'[t]$ for some $a_i \in K'$. By Lemma 2.1, all a_i's are conjugates of b and hence are conjugates of a. But then the previous discussion tells us that a is the only conjugate of itself lying in K', so every a_i is in fact a and $f(t) = (t - a)^n \in K[t]$.

Let $g(t) = (t - a)^{n-2}$. By definition of $f(t)$, we have $g(b) \neq 0$. By [L: Theorem 2], $g(b)bg(b)^{-1}$ satisfies the equation $(t - a)^2 = 0$. If $g(b)bg(b)^{-1} = a$, then $ag(b) - g(b)b = 0$. So b satisfies the equation $ag(t) - g(t)t = 0$ for some $g(t)t - ag(t) \neq 0$ in $K[t]$. This contradicts the minimality of f. Therefore $g(b)bg(b)^{-1} \neq a$ and it satisfies the equation $t^2 - 2at + a^2 = 0$. Since a, b, $g(b)$ are in D, by Lemma 1.2, we conclude $(g(b)bg(b)^{-1} - a)^{-1}$ is not algebraic over K. This contradicts the assumption D is algebraic over K. ■

THEOREM 4.6. *Let D', D, K be as before. If there exist $a_1, \ldots, a_n \in K$ such that*

$$C(a_1, \ldots, a_n) := \{y \in D : ya_i = a_i y,\ 1 \leqq i \leqq n\} = K,$$

then $D = K$.

PROOF. We prove this by induction on n. If $n = 1$, then we have $C(a_1) = K$ and $C'(a_1) = K'$. By Lemma 4.5, $a_1 \in Z(D)$. Therefore, $D = C(a_1) = K$.

Suppose $C(a_1, \ldots, a_{k+1}) = K$ for some $a_1, \ldots, a_{k+1} \in K$. Consider $D_1 := C(a_1)$. If $D_1 = K$, then we are done. So, in particular, we may assume $K \subset D_1$. Clearly, D_1 is algebraic over K, as by assumption D is algebraic over K. In D_1, it is easy to see that

$$C_1(a_2, \ldots, a_{k+1}) = \{y \in D_1 : ya_i = a_i y, 2 \leqq i \leqq k+1\} = C(a_1, \ldots, a_{k+1}) = K.$$

Therefore, by induction, we see that $D_1 = K$. This contradicts the assumption that $K \neq D_1$. ■

We now apply the previous theorem to some specific cases. Namely,we can dispose of the case when tr.d. $K/Z(D)$ is finite. In particular, we have the following:

COROLLARY 4.7. *Let D, K be as before. Then* tr.d. $K/Z(D)$ *is finite iff D is a field. In particular, D is a field if* tr.d. $K/\mathbf{Q}$ *is finite.*

PROOF. Obviously, we only need to prove sufficiency. Let $\{x_1, \ldots, x_n\}$ be a transcendence base of K over $Z(D)$. Then $C(x_1, \ldots, x_n)$ is an ordered division subring in D. Moreover, every element of K is algebraic over $Z(D)(x_1, \ldots, x_n)$. Therefore by Albert's Theorem, K is central in $C(x_1, \ldots, x_n)$. As K is a maximal subfield, it forces $C(x_1, \ldots, x_n) = K$. So the corollary follows from Theorem 4.6. ■

5. Albert's Theorem in $D[i]$

Let (D, P) be an ordered division ring. In this section, we do not assume D is algebraic over a maximal subfield. Let $D[i]$ be as defined in Section 4. In view of that section, it seems worthwhile to take a closer look at the division ring $D[i]$. Firstly, observe that the center of $D[i]$ is clearly $Z(D)[i]$. Our goal is to show that Albert's Theorem is also true in $D[i]$. In other words, $Z(D)[i]$ is algebraically closed in $D[i]$. For convenience, we shall denote the natural valuation of (D, P) by v.

LEMMA 5.1. *Let $x, y \in D$. For any $0 \neq a + bi \in D[i]$, we let $x' + y'i := (a + bi)(x + yi)(a + bi)^{-1}$. If $v(x) = v(y)$ and $x > 0$, then $x' > 0$.*

PROOF. Recall that in $D[i]$, $(a + bi)^{-1} = (1 - a^{-1}bi)(a + ba^{-1}b)^{-1}$. Hence

$$x' = [(ax - by) + (bx + ay)a^{-1}b](a + ba^{-1}b)^{-1}.$$

Observe also that as v is compatible with the fixed ordering on D, we have for any $c, d \in D^*$ with $v(c) < v(d)$, $c + d > 0$ if and only if $c > 0$.

Case (i). Either a or b is 0.

Since i is central, we may assume $a \neq 0$. In this case, we have $x' = axa^{-1}$ which is clearly positive.

Case (ii). Both a and b are nonzero and $v(a) < v(b)$.

Clearly, we have $v(ax) < v(by)$. So $v(ax) < v(aya^{-1}b) < v(bxa^{-1}b)$ and $v(ax) < v(-by + (bx + ay)a^{-1}b)$. Thus $x' > 0$ iff $ax(a + ba^{-1}b)^{-1} > 0$. But obviously, a, $(a + ba^{-1}b)$ are of the same sign in D. As $x > 0$, it follows that $x' > 0$.

Case (iii). Both a and b are nonzero and $v(a) > v(b)$.

Multiplying $(a + bi)$ by i, we go back to Case (ii) and we are done.

Case (iv). Both a and b are nonzero and $v(a) = v(b)$.

Since ax, $bxa^{-1}b$ are of the same sign and $(ax + bxa^{-1}b)(a + ba^{-1}b)^{-1} > 0$. To show $x' > 0$, it suffices to check that $v(aya^{-1}b - by) > v(ax)$. But clearly,

$$v(aya^{-1}b - by) = v(a) + v(ya^{-1}b - a^{-1}by)$$
$$= v(a) + v(ya^{-1}by^{-1} - a^{-1}b) + v(y).$$

As $v(a) = v(b)$, we have $a^{-1}b \in U_v$. By [Sch: Lemma 1.12], we have $v(ya^{-1}by^{-1} - a^{-1}b) > 0$. It follows that $v(aya^{-1}b - by) > v(a) + v(y) = v(ax)$. ■

Lastly, we shall prove that $Z(D)[i]$ is algebraically closed in $D[i]$.

THEOREM 5.2. *Let $x + yi \in D[i]$ be algebraic over the center $Z(D)[i]$. Then $x + yi \in Z(D)[i]$.*

PROOF. Let $x + yi$ be as assumed above. Note that if either x or y is 0, say $y = 0$, then x is already algebraic over $Z(D)$ in D. Albert's result then implies x being central. Thus we may assume both x and y are nonzero.

Let $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ be the minimal polynomial of $x + yi$ over $Z(D)[i]$. If $x + yi \notin Z(D)[i]$, then we may assume $n \geqq 2$ and $a_{n-1} = 0$. Replacing $x + yi$ by $(1 + i)(x + yi) = (x - y) + (x + y)i$ if necessary, we can also assume $v(x) = v(y)$. (Note that the minimal monic polynomial of $(x - y) + (x + y)i$ over $Z(D)[i]$ is $t^n + a_{n-2}(1 + i)^2 t^{n-2} + \cdots + (1 + i)^n a_0$.) Lastly, we may of course assume $x > 0$.

By Lemma 2.1, there exists $a_j + b_j i \in D[i]$, $j = 1, 2, \ldots, n$ such that

$$0 = \sum_{j=1}^{n} (x_j + y_j i) \qquad \text{where } x_j + y_j i = (a_j + b_j i)(x + yi)(a_j + b_j i)^{-1}.$$

By Lemma 5.1, $x_j > 0$ for $j = 1, 2, \ldots, n$. This is clearly a contradiction. ■

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